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EQUATION FOR THE STRUCTURE FUNCTION OF A TURBULENT STATIONARY ISOTROPIC VELOCITY FIELD AND ITS SOLUTION IN THE INERTIAL SCALE INTERVAL

V. A. Sosinovich

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A closed equation is obtained for the structure function of a turbulent stationary isotropic velocity field and the equation is solved in the inertial scale interval.

1. A closed equation for the structure function of a turbulent isotropic nonstationary velocity field is obtained in [1, 2]. In this paper, we attempt to obtain the stationary form of this equation and solve it in the inertial scale interval.

In order to obtain the stationary form of Eq. (22) in [2] for the structure function D(r), it is necessary to pass in this equation to the limit $t > \infty$ and calculate the integral over the time variable. However, in so doing, it is necessary to take into account the fact that the integrand on the right side of this equation depends on time explicitly and through the function being sought. The temporal dependence of the function $D(r, \tau)$ on the right side of the equation, generally speaking, cannot be neglected, since the integral over τ is calculated from $\tau = 0$ to $t \to \infty$. It is clear that for τ close to zero, the functions sought depend strongly on time. However, when some certain conditions are satisfied, this dependence can be neglected and the integration over τ can be carried out. We will obtain these conditions.

One of the terms on the right side of Eq. (22) in [2] can be written in the form

$$I_t \infty \int\limits_0^\infty d\rho \int\limits_0^t \frac{d\tau}{(t-\tau)^3} k_0^1(\rho,\ n,\ q)\,\varphi(\rho,\ \tau), \tag{1}$$

where k_0^1 is defined by Eq. (28) in [2]; here and in what follows, we will understand the symbol φ to mean the part of the integrand which depends on τ through the functions sought and does not depend on τ explicitly. Changing the variable of integration τ according to the equation $(t - \tau)^{-1} = z$ and using Eq. (28) in [2] for k_0^1,

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A. V. Lykov Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk.
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we obtain

$$I_t \infty \int_0^\infty d\rho \sum_{m=1}^{3} \int_{1/t}^{\infty} dz z\psi(\rho,\ n,\ z) z^{2m-2} \varphi(\rho,\ t-z^{-1}) + \ldots \tag{2}$$

Here, only one of a series of terms, all of which can be analyzed in a completely analogous manner, is written out.

We substitute in (2) the expression for ψ in the form (29) in [2] and find

$$I_t \infty \int_0^\infty d\rho \sum_{m=1}^{3} \int_{1/t}^{\infty} dz \exp[z^2\Lambda^2(n^2-1)(\rho^2 n^2-1)] \times \{\Phi[z\Lambda(1-\rho n^2)] + \Phi[z\Lambda(1+\rho n^2)]\} z^{2m-2}\varphi(\rho,\ t-z^{-1}), \tag{3}$$

$\Lambda = r/\sqrt{2n}D(\rho, r)$; and $\Phi(x)$ is the probability integral. If we use the expansion of the function $\Phi(x)$ in a series (see Eq. 8.253(1) in [3]), then the expression for I_t can be represented in the form

$$I_t \infty \int_0^\infty d\rho \sum_{m=1}^{3} \sum_{k=0}^{\infty} \int_{1/t}^{\infty} dz z^{2(m+k)-1} \{\exp[-z^2\Lambda^2 n^2(1-\rho)^2] \times \varphi_1(\rho,\ t-z^{-1}) + \exp[-z^2\Lambda^2 n^2(1+\rho)^2]\varphi_2(\rho,\ t-z^{-1})\}. \tag{4}$$

The meaning of the functions φ_1 and φ_2 in (4) is the same as that of φ in Eqs. (1)-(3).

We assumed that there exists a relaxation time t_r such that if $\tau > t_r$, then the function sought $D(r, \tau)$ does not depend on the argument τ. The condition that the functions of z in (4) be stationary has the form

$$t > t_r + z^{-1}. \tag{5}$$

The initial, i.e., strongly depending on the time argument, values of the functions are realized for $t - z^{-1} \to 0$. If, in this case, $t \to \infty$, then $z \to 0$. However, only the values of functions for which $z = z_m \equiv \left(m+k-\frac{1}{2}\right)^{\frac{1}{2}}/\alpha_{(\pm)}$, where $\alpha_{(\pm)} = \Lambda n(1 \pm \rho)$, give the main contribution to the integral over z. This follows from the fact that the function $z^{2(m+k)-1}e^{-z^2\alpha^2(\pm)}$ has a maximum for the value $z = z_m$. But, if $z = z_m \neq 0$, then it is always possible to choose t such that the steady state condition (5) be satisfied for $z = z_m$. Indeed, the right side in the equality

$$t > t_r + \alpha_{(\pm)}/\left(m+k-\frac{1}{2}\right) \tag{6}$$

is always finite is a bounded flow, since

$$\alpha_{(\pm)} \infty \frac{rL}{\bar{v^2}^{1/2}}, \tag{7}$$

where L is the size of the system; $\bar{v^2}$ is the mean-square value of the fluctuating velocity.

If the flow is infinite, then

$$\alpha_{(\pm)} \sim \frac{rL_r}{D^{1/2}(L_r)}, \tag{8}$$

where L_r is the maximum size of vortices which have a direct effect on vortices with scale r. The existence of $L_r < \infty$ is the second assumption necessary for condition (6) to be satisfied. Thus, two assumptions are important: a) the existence of a relaxation time $t_r < \infty$; and b) quasilocal nature of the interaction of fluctuations of different scales: $L_r < \infty$. Under these assumptions, the dependence of the right side of Eq. (23) in [2] on τ can be neglected for $t \gg t_r$ and, setting $t \to \infty$, the integration over the variable τ can be carried out.

In going over to the steady-state form for the equation for D(r), we omit all terms proportional to derivatives with respect to time and assume that in this case the following relation is valid [4]:

$$\varepsilon_d = \lim_{\rho\to 0} \frac{15}{2}\frac{\nu}{r^2} D''_{\rho\rho}(\rho r) = \varepsilon,$$

i.e., the magnitude of the energy dissipation becomes equal to the magnitude of the pumping.

The equation for D(r) takes the form

$$D(r) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\rho \frac{\rho^2}{D^{1/2}(\rho r)} \mathscr{D}(\rho,\ r). \tag{9}$$

Here

$$\mathscr{D}(\rho, r) = \sum_{l=1}^{2} k_0^l(\rho,n)\left[\frac{2\nu}{r} D''_{\rho\rho}(\rho, r)\eta_{l_1} + 2\varepsilon r\left(\frac{r}{L}\right)^2 \rho^2\eta_{l_2} + \frac{4}{15}\varepsilon r\eta_{l_3}\right] + \frac{1}{8D^{1/2}(\rho r)}\sum_{j=1}^{3}\Big\{-\sqrt{2\pi}\,k_j(\rho, n) +$$

$$+\frac{1}{6D^{3/2}(\rho r)}\sum_{l=1}^{2} k_j^l(\rho, n)\left[\frac{\nu}{r} D''_{\rho\rho}(\rho r)\tau_{l_1} + \varepsilon r\left(\frac{r}{L}\right)^2 \rho^2\tau_{l_2} + \frac{2}{15}\varepsilon r\tau_{l_3}\right]\Big\} M_j(\rho, r). \tag{10}$$

In Eq. (10) $M_j(\rho, r)$, η_{l_p}, and τ_{l_r} are interpreted according to Eqs. (24) and (25) in [2]. The following new notation is introduced:

$$k_j^l = \lim_{t\to\infty} 2n^2\Lambda^2 \int_0^t \frac{d\tau}{(t-\tau)^3}\, k_j^l(\rho, n, q),\ j = 0, 1, 2, 3;\ l = 1, 2, \tag{11}$$

$$k_j = \lim_{t\to\infty} \frac{2n^2\Lambda^2}{\sqrt{2\pi}} \int_0^t \frac{d\tau}{(t-\tau)^3}\, k_j(\rho, n, q),\ j = 1, 2, 3. \tag{12}$$

The quantities $k_j^l(\rho, n, q)$ and $k_j(\rho, n, q)$ are defined by Eqs. (28) and (30) in [2]. Substituting these expressions into (11) and (12) and replacing the variable of integration using the equation $\tau = -\Lambda z^{-1} + t$, we obtain the following equations for calculating k_j^l and k_j:

$$k_j^l = (n^2-1)\sum_{k=0}^{5}\rho^{2k}\sum_{m=1}^{5}\left[a_{jk}^{lm}\psi_{2m-1} + b_{jk}^{lm}\psi_{2m-1}^{(c)} + \rho c_{jk}^{lm}\psi_{2m-1}^{(s)}\right], \tag{13}$$

$$k_j = \frac{n(n^2-1)}{\sqrt{\pi}}\sum_{k=0}^{3}\rho^{2k}\sum_{m=1}^{3}\left[a_{jk}^{m}\psi_{2m} + \rho^{-1}b_{jk}^{m}\psi_{2m-2}^{(s)} + c_{jk}^{m}\psi_{2m-2}^{(c)}\right]. \tag{14}$$

Here, the following notation is used:

$$\psi_l = 2\int_0^\infty dz z^l \psi(\rho, n, z); \tag{15}$$

$$\begin{vmatrix}\psi_l^{(c)}\\ \psi_l^{(s)}\end{vmatrix} = 2\int_0^\infty dz z^l \exp\left[-z^2n^2(1+\rho^2)\right]\begin{vmatrix}\mathrm{ch}\,(2z^2n^2\rho)\\ \mathrm{sh}\,(2z^2n^2\rho)\end{vmatrix}, \tag{16}$$

where the expression for $\psi(\rho, n, z)$ is defined by Eq. (29) in [2]. Calculation gives the following results:

$$\psi_1 = \mu^{1/2}\,\mathrm{arctg}\frac{\mu^{1/2}}{\frac{1}{2}n^2(1+\rho^2) - 1}, \tag{17}$$

$$\psi_2 = \begin{cases}\dfrac{\sqrt{\pi}}{n(n^2-1)(1-\rho^2)}, & \text{if } \dfrac{1}{n^2} < \rho < 1,\\[2ex] -\dfrac{\sqrt{\pi}\rho}{n(\rho^2n^2-1)(1-\rho^2)}, & \text{if } \rho < \dfrac{1}{n^2} \text{ or } \rho > 1.\end{cases} \tag{18}$$

The values of the integrals ψ_l for other values of l are calculated from the recurrence equation

$$\psi_{l+2} = -\frac{\Gamma\left(\frac{l+3}{2}\right)}{\mu(l+1)}\left\{\frac{1}{[n|1-\rho^2|]^{l+1}}\left[\frac{|\rho n^2-1|^{l+1}(1+\rho)^{l+1}}{(\rho n^2-1)^l} - (\rho n^2+1)|1-\rho|^{l+1}\right] - \frac{l}{\Gamma\left(\frac{l+1}{2}\right)}\psi_l\right\}. \tag{19}$$

Here $\Gamma(x)$ is the gamma function;

$$\mu = (n^2-1)(1-\rho^2n^2); \tag{20}$$

$$\psi_l^{\binom{c}{s}} = \frac{\Gamma\left(\frac{l+1}{2}\right)\left[(1+\rho)^{l+1} \pm |1-\rho|^{l+1}\right]}{2^{n+l}|1-\rho^2|^{l+1}}. \tag{21}$$

Using these results and the expressions for the coefficients a, b, and c presented in Appendix A in [2], it is possible to obtain expressions for k_j^l and k_j:

$$k_j^l = \frac{1}{d^3}\left[a_j^l(\rho,\ n) + b_j^l(\rho,\ n)\,\psi(\rho,\ n)\right], \tag{22}$$

where

$$\psi(\rho,\ n) = \begin{cases} \dfrac{1-\rho^2}{\sqrt{-md}}\operatorname{arctg}\dfrac{\sqrt{-md}}{\frac{1}{2}n^2(1+\rho^2)-1}, & \text{if } \rho < \dfrac{1}{n};\\[2ex] 2(1-\rho^2)/m, & \text{if } \rho = \dfrac{1}{n};\\[2ex] \dfrac{1-\rho^2}{\sqrt{md}}\operatorname{arth}\dfrac{\sqrt{md}}{\frac{1}{2}n^2(1+\rho^2)-1}, & \text{if } \rho > \dfrac{1}{n};\end{cases} \tag{23}$$

$$d = \rho^2 n^2 - 1;\ m = n^2 - 1.$$

The matrices $a_j^e(\rho, n)$ and $b_j^e(\rho, n)$ are some polynomials in powers of ρ^2 with coefficients depending on n^2:

$$a_0^0 = d^3;\ b_0^0 = \frac{1}{2}d^3n^2;\ a_0^1 = d^2m(-1+\rho^2);\ b_0^1 = d^2m\left(-1+\frac{n^2}{2}+\right.$$

$$\left.+\rho^2\frac{n^2}{2}\right);\ a_0^2 = -d^2m\rho^2;\ b_0^2 = d^2m\left(1-\rho^2\frac{n^2}{2}\right);\ a_1^0 = d\frac{m^2}{n^2}(-2+$$

$$+3n^2-\rho^2n^2);\ b_1^0 = d\frac{m^2}{2}(4-3n^2-\rho^2n^2);\ a_0^2 = d^2\frac{m^2}{n^2};\ b_0^2 = -d^2\frac{m^2}{2};$$

$$a_3^0 = d\frac{m}{n^2}\left[\left(-3+2n^2+3n^4\right)+\rho^2(3n^2-5n^4)\right];\ b_3^0 = \frac{3}{2}dm^2(-1-n^2+$$

$$+\rho^2n^2);\ a_1^1 = \frac{m^3}{n^2}(2-15n^2+13\rho^2n^2);\ b_1^1 = \frac{3}{2}m^3(-4+5n^2-\rho^2n^2);$$

$$a_2^1 = \frac{d}{n^2}[(-11+25n^2+7n^4-3n^6)+\rho^2(20n^2-58n^4+2n^6)+$$

$$+\rho^4(-6n^4+24n^6)];\ b_2^1 = \frac{3}{2}d[(-1+9n^2-3n^4+n^6)+\rho^2(2n^2-$$

$$-14n^4)+\rho^4(-2n^4+8n^6)];\ a_3^1 = \frac{1}{n^2}[(-3+9n^2+11n^4-15n^6)+$$

$$+\rho^2(-3n^2-20n^4+19n^6)+\rho^4(6n^4-4n^6)];\ b_3^1 = \frac{3}{2}m^2(3+5n^2-3\rho^2n^2);$$

$$a_1^2 = \frac{3}{n^6}[(-24+60n^2-28n^4+10n^6-3n^8)+\rho^2(48n^2-112n^4-$$

$$-5n^6-36n^8-12n^{10})+\rho^4(-42n^4+115n^6-27n^8-n^{10})+$$

$$+\rho^6(18n^6-48n^8+15n^{10})];\ b_1^2 = \frac{9}{2n^2}[(4+4n^2-4n^4+n^6)+$$

$$+\rho^2(12-46n^2+29n^4-14n^6+4n^8)+\rho^4(-18n^2+53n^4-23n^6+$$

$$+3n^8)+\rho^6(6n^4-16n^6+5n^8)];\ a_2^2 = \frac{3dm}{n^4}[-2+$$

$$+\rho^2(5n^2-3n^4)];\ b_2^2 = \frac{9}{2}dm^2\rho^2;\ a_3^2 = \frac{3}{n^4}[(6-6n^2+8n^4-6n^6)+$$

$$+\rho^2(-21n^2+19n^4+7n^6-9n^8)+\rho^4(15n^4-28n^6+15n^8)];$$

$$b_3^2=\frac{9}{2}m^2[2+\rho^2(3+3n^2)-3\rho^4n^2]; \tag{24}$$

$$k_j(\rho,\ n)=\frac{m}{2n^2d}\begin{cases}\dfrac{1-\rho n^2}{(1-\rho)^3}k_j^{<}(\rho,\ n), & \text{if } \rho<\dfrac{1}{n^2};\\ 0\quad, & \text{if } \dfrac{1}{n^2}<\rho<1;\\ \rho^{-1}k_j^{>}(\rho,\ n)\quad, & \text{if } \rho>1;\end{cases} \tag{25}$$

$$k_1^{<}(\rho,\ n)=\frac{m^2}{n^2d(1-\rho)^2}\{[-3+3n^2+\rho^2(11n^2-3n^4)+$$
$$+\rho^4(-n^2+11n^4+2n^6)+\rho^6(6n^4-n^6+3n^8)+\rho^8(-3n^6-6n^8-3n^{10})]+$$
$$+\rho[-6n^2+\rho^2(2n^2+2n^4)+\rho^4(-22n^4-16n^6)+\rho^6(12n^6+12n^8)]\};$$
$$k_1^{>}(\rho,\ n)=\frac{m^2}{d};\quad k_2^{<}(\rho,\ n)=m\{[1-\rho^2n^2+\rho^4(n^2+n^4)]+\rho[-2\rho^2n^2]\};$$
$$k_2^{>}(\rho,\ n)=m;\quad k_3^{<}(\rho,\ n)=\frac{1}{d(1-\rho)^2}\{[(-3+3n^4)+\rho^2(-3+9n^2-n^4- \tag{26}$$
$$-3n^6)+\rho^4(-9n^2-12n^4+29n^6+2n^8)+\rho^6(-3n^2+10n^4-2n^6-18n^8+3n^{10})+$$
$$+\rho^8(3n^4-2n^8-3n^{12})]+\rho[(6-6n^4)+\rho^2(-6n^2-4n^4+2n^6)+$$
$$+\rho^4(12n^2+4n^4-16n^8)+\rho^6(-12n^4+4n^6+4n^8+12n^{10})]\};$$
$$k_3^{>}(\rho,\ n)=\frac{1}{d}[(3-3n^2-2n^4)+\rho^2(-3n^2+5n^4)].$$

II. Let us proceed to study Eqs. (9) and (10) in the inertial scale interval. The inertial interval is determined by the condition [4] $\eta \ll r \ll L$, where L is the outer scale and η is the Kolmogorov turbulence scale.

In the inertial interval, the terms $\sim \nu$ and $\sim\left(\frac{r}{L}\right)^2$ drop out of Eqs. (9) and (10). The first terms are related to the direct action of viscous forces and the second are related to the direct action of large vortices on vortices with scale size r. According to the definition of the inertial interval, both are small. Taking this into account, as well as Eqs. (25) in [2], we obtain the equation for the structure function in the inertial interval

$$D(r)=\frac{1}{\sqrt{2\pi}}\int_0^\infty d\rho\frac{\rho^2}{D^{1/2}(\rho r)}\left\{k_0^2(\rho,\ n)\frac{4}{3}\varepsilon r+\frac{1}{8D^{1/2}(\rho r)}\times\right.$$
$$\left.\times\sum_{j=1}^{3}\left[-\sqrt{2\pi}\,k_j(\rho,\ n)+\frac{k_j^2(\rho,\ n)}{6D^{3/2}(\rho r)}\frac{4}{15}\varepsilon r\right]M_j(\rho r)\right\}. \tag{27}$$

Here $M_j(\rho r)$, j = 1, 2, 3, are defined by Eq. (24) in [2].

In solving (27), it turns out that the integral over ρ on the right side of this equation diverges at infinity. This is related to the fact that the right side of this equation contains an infinite constant. In order to eliminate it, it is enough to differentiate the left and right sides of this equation with respect to r and then to solve the resulting differential equation. For convenience in differentiating, we rewrite Eq. (27) in the form

$$D(r)=\frac{1}{\sqrt{2\pi}}\int_0^\infty dy\,\frac{y^2}{D^{1/2}(y)}\left\{\frac{1}{r^2}\,k_0^2\left(\frac{y}{r},\,n\right)\frac{4}{3}\,\varepsilon-\frac{1}{8D^{1/2}(y)}\times\right.$$
$$\left.\times\sum_{j=1}^{3}\left[-\frac{\sqrt{2\pi}}{r^3}\,k_j\left(\frac{y}{r},\,n\right)+\frac{2\varepsilon}{45D^{3/2}(y)}\,\frac{1}{r^2}\,k_j^2\left(\frac{y}{r},\,n\right)\right]M_j(y)\right\}. \tag{28}$$

Differentiating the left and right sides of (28) with respect to r and introducing the definition

$$P(r)=\frac{1}{2}\,D_r'(r), \tag{29}$$

we obtain

$$P(r)=\frac{1}{\sqrt{2\pi}\,r^2}\int_0^\infty dy\,\frac{y}{D^{1/2}(y)}\left\{-\frac{2}{3}\,\varepsilon G_0^2\left(\frac{y}{r},\,n\right)+\frac{1}{16D^{1/2}(y)}\times\right.$$
$$\left.\times\sum_{j=1}^{3}\left[\frac{\sqrt{2\pi}}{y}\,G_j\left(\frac{y}{r},\,n\right)-\frac{2\varepsilon}{45D^{3/2}(y)}\,G_j^2\left(\frac{y}{r},\,n\right)\right]M_j(y)\right\}, \tag{30}$$

where

$$G_j^2\left(\frac{y}{r},\,n\right)=-r^2y\,\frac{d}{dr}\left[\frac{1}{r^2}\,k_0^2\left(\frac{y}{r},\,n\right)\right]=\frac{d}{d\rho}\,[\rho^2k_0^2(\rho,\,n)]_{|\rho=\frac{y}{r}}; \tag{31}$$

$$G_j\left(\frac{y}{r},\,n\right)=-r^2y^2\,\frac{d}{dr}\left[\frac{1}{r^3}\,k_j\left(\frac{y}{r},\,n\right)\right]=\frac{d}{d\rho}\,[\rho^3k_j(\rho,\,n)]_{|\rho=\frac{y}{r}}; \tag{32}$$

$$M_j(y)=4\int_0^\infty dxxP(x)\left\{\left[a^j\left(\frac{x}{y}\right)P(x)-b^j\left(\frac{x}{y}\right)xP'(x)\right]+\right.$$
$$\left.+\frac{1}{32y^2}\sum_{n=1}^{4}\sum_{k=-6}^{7}\gamma_k^{jn}\left(\frac{x}{y}\right)\int_{|y-x|}^{y+x}dzzP(z)\,\beta_n(x,\,z)\left(\frac{z}{y}\right)^k\right\}. \tag{33}$$

Equation (30) can be viewed as an equation for the function P(r) in the inertial scale interval. Such a function was first examined in [5]. In so doing, an argument was given in [5] supporting the fact that it is this function that can be the object of the theory of turbulence in the universal scale interval. In the inertial scale interval, the functions P(r) and D(r) in Eq. (30) must have the following form:

$$P(r)=\frac{1}{3}\,C\varepsilon^{2/3}r^{-1/3}, \tag{34}$$

$$D(r)=C\varepsilon^{2/3}r^{2/3}. \tag{35}$$

This follows from dimensional analysis. The next problem is to determine the values of the constant C in Eqs. (34) and (35). Substituting P(r) and D(r) in the form (34) and (35) into Eq. (30), we obtain

$$\frac{1}{3}\,C\varepsilon^{2/3}r^{-1/3}=\frac{1}{\sqrt{2\pi}\,r^2}\int_0^\infty dy\,\frac{y}{C^{1/2}\varepsilon^{1/3}y^{1/3}}\left\{-\frac{2}{3}\,\varepsilon G_0^2\left(\frac{y}{r},\,2\right)+\right.$$
$$\left.+\frac{1}{16C^{1/2}\varepsilon^{1/3}y^{1/3}}\sum_{j=1}^{3}\left[\frac{\sqrt{2\pi}}{y}\,G_j\left(\frac{y}{r},\,2\right)-\frac{2\varepsilon}{45C^{3/2}\varepsilon y}\,G_j^2\left(\frac{y}{r},\,2\right)\right]\tilde{M}_j(y)\right\}. \tag{36}$$

In writing (36), we took into account the fact that in the inertial interval

$$\tilde{n}=\sqrt{1+\frac{\tilde{D}(y)}{y\tilde{P}(y)}}=2. \tag{37}$$

The expression for $\tilde{M}_j(y)$ in the inertial interval has the form

$$\tilde{M}_j(y)=\frac{4}{9}\,C^2\varepsilon^{4/3}y^{4/3}I_j, \tag{38}$$

where the constants I_j are defined by the following expression:

$$I_j = \int_0^\infty dx \left\{ x^{1/3} \left[a^j(x) + \frac{1}{3} b^j(x) \right] + \frac{x^{2/3}}{32} \sum_{k=-6}^{7} \frac{\alpha_k^j(x)}{k+5/3} \left[(1+x)^{k+5/3} - |1-x|^{k+5/3} \right] \right.. \quad (39)$$

In the last equality, we used the notation

$$\alpha_k^j(x) = \sum_{n=1}^{4} \gamma_k^{jn}(x)\tilde{\beta}_n. \quad (40)$$

The matrix β, which is defined by Eq. (23) in [1], has the following form in the inertial interval:

$$\tilde{\beta}_n = \begin{vmatrix} 1/9 \\ 1/3 \\ 1/3 \\ 1 \end{vmatrix}. \quad (41)$$

A calculation, making use of Appendix B in [2], leads to the following result:

$$I_j \approx \begin{vmatrix} -2 \\ 15 \\ 7 \end{vmatrix}. \quad (42)$$

Substituting in (36) the expression for $\tilde{M}_j(y)$ in the form (38), dividing both sides of the equation by $\frac{1}{3}\varepsilon^{2/3}r^{2/3}$, we obtain

$$C = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\rho \rho^{2/3} \left\{ -\frac{2}{C^{1/2}} G_0^2(\rho, 2) + \frac{\sqrt{2\pi}\, C}{12} \sum_{j=1}^{3} I_j G_j(\rho, 2) - \frac{1}{270\, C^{1/2}} \sum_{j=1}^{3} I_j G_j^2(\rho, 2) \right\}. \quad (43)$$

We introduce the following notation:

$$L_j = -\int_0^\infty d\rho \rho^{2/3} G_j(\rho, 2), \quad j = 0, 1, 2, 3, \quad (44)$$

$$K_j = \int_0^\infty d\rho \rho^{2/3} G_j(\rho, 2), \quad j = 1, 2, 3. \quad (45)$$

With this notation, equality (43) can be rewritten in the form

$$C^{3/2} \left[1 - \frac{1}{12} \sum_{j=1}^{3} I_j K_j \right] = \frac{1}{\sqrt{2\pi}} \left[2L_0 + \frac{1}{270} \sum_{j=1}^{3} I_j L_j \right]. \quad (46)$$

From (46), we obtain the equation for calculating the constant C:

$$C = \left\{ \frac{\frac{1}{\sqrt{2\pi}} \left[2L_0 + \frac{1}{270} \sum_{j=1}^{3} I_j L_j \right]}{1 - \frac{1}{12} \sum_{j=1}^{3} I_j K_j} \right\}^{2/3}. \quad (47)$$

In order to make use of this equation, it is necessary to know the value of the integrals (44) and (45). The functions G_j^2 and G_j in the integrand can be calculated from Eqs. (31) and (32). The results are:

$$G_0^2(\rho, 2) = \frac{6\rho}{(4\rho^2-1)^2} \left[(3\rho^2 - 6\rho^4) + \frac{-1+4\rho^2-12\rho^4+12\rho^6}{1-\rho^2} \psi(\rho, 2) \right],$$

$$\tilde{G}_1^2(\rho, 2) = \frac{9\rho}{(4\rho^2-1)^4} \left[\frac{15}{4} + 139\rho^2 + 352\rho^4 - 158\rho^6 + 264\rho^8 + \right.$$

$$\left. + \frac{1}{1-\rho^2} (-5 - 250\rho^2 - 339\rho^4 - 106\rho^6 + 580\rho^8 - 528\rho^{10}) \psi(\rho, 2) \right];$$

$$G_2^2(\rho,\ 2)=\frac{9\rho}{8(4\rho^2-1)^3}\Big[2+64\rho^2+72\rho^4+$$

$$+\frac{1}{1-\rho^2}(-144\rho^2+72\rho^4-144\rho^6)\,\psi(\rho,\ 2)\Big],$$

$$G_3^2(\rho,\ 2)=\frac{3\rho}{(4\rho^2-1)^4}\Big[\frac{137}{4}+737\rho^2+365\rho^4-324\rho^6+ \qquad (48)$$

$$+\frac{1}{1-\rho^2}(-54-1242\rho^2+81\rho^4+162\rho^6+648\rho^8)\,\psi(\rho,\ 2)\Big].$$

Here, the function $\psi(\rho, 2)$ is defined by Eq. (23) with $n = 2$. If we use the expansion for the function $\psi(\rho, 2)$ around the point $\rho = 1/2$ in the form

$$\lim_{\rho\to\frac{1}{2}}\psi(\rho)=\frac{2}{3}(1-\rho^2)\Big[1+\frac{1}{9}(4\rho^2-1)+\frac{1}{45}(4\rho^2-1)^2+\frac{1}{189}(4\rho^2-1)^3+\ldots\Big], \qquad (49)$$

then it can be shown that the functions $G_j^2(\rho, 2)$ do not have singularities at the point $\rho = 1/2$.

Using the expansion of the function $\psi(\rho, 2)$ in a series for $\rho \to \infty$

$$\lim_{\rho\to\infty}\psi(\rho,\ 2)=(1-\rho^2)\Big[\frac{1}{2}\,\frac{1}{\rho^2}+\frac{1}{4}\,\frac{1}{\rho^4}+\ldots\Big], \qquad (50)$$

it can be shown that for $\rho \to\infty$ all functions $G_j^2(\rho, 2)$ have the form

$$\lim_{\rho\to\infty}G_j(\rho,\ 2)=\frac{A_j}{\rho^3}+\frac{B_j}{\rho^5}. \qquad (51)$$

A study of the functions $G_j^2(\rho, 2)$ for $\rho \to 1$ shows that they all have at this point an integrable singularity

$$\lim_{\rho\to 1}G_j^2(\rho,\ 2)\sim\ln|1-\rho|. \qquad (52)$$

The expression for the functions $G_j(\rho, 2)$, $j = 1, 2$, and 3 has the form

$$G_j(\rho,\ 2)=\begin{cases}G_j^{<}, & \text{if}\quad \rho<\frac{1}{4},\\ 0, & \text{if}\quad \frac{1}{4}<\rho<1,\\ G_j^{>}, & \text{if}\quad \rho>1,\end{cases} \qquad (53)$$

where

$$G_1^{<}=\frac{27\rho^2}{32(1-\rho)^6(4\rho^2-1)^3}[-27-436\rho^2-1756\rho^4-62112\rho^6+$$

$$+340736\rho^8-432640\rho^{10}-230400\rho^{12}+\rho(222-120\rho^2+$$

$$+22808\rho^4-21280\rho^6-336000\rho^8+775680\rho^{10})];$$

$$G_1^{>}=\frac{27(-\rho-4\rho^3)}{4(4\rho^2-1)^3};$$

$$G_2^{<}=\frac{9\rho^2}{8(1-\rho)^4(4\rho^2-1)^2}[-3+20\rho^2-404\rho^4+608\rho^6+$$

$$+960\rho^8+\rho(16-80\rho^2+976\rho^4-2336\rho^6)];\quad G_2^{>}=\frac{-9\rho}{4(4\rho^2-1)^2};$$

$$G_3^{<} = \frac{3\rho^2}{8(1-\rho)^6(4\rho^2-1)^3}[-135-835\rho^2-17780\rho^4-$$

$$-134772\rho^6+1064240\rho^8-1575872\rho^{10}-3287296\rho^{12}+\rho(990-$$

$$-3360\rho^2+113192\rho^4-321584\rho^6-543072\rho^8+2162304\rho^{10})];$$

$$G_3^{>} = \frac{3(41\rho+28\rho^3)}{4(4\rho^2-1)^3}.$$

It is easy to see that for $\rho \to \infty$ the expansion of the functions $G_j(\rho, 2)$ has the form

$$\lim_{\rho\to\infty} G_j(\rho,\ 2) = \frac{G_j}{\rho^3} + \frac{D_j}{\rho^5} + \ldots \tag{54}$$

From the properties of the functions $G_j^2(\rho, 2)$ and $G_j(\rho, 2)$ mentioned above, it is evident that the integrals L_j and K_j can be calculated.

As a result of a numerical integration using Eqs. (44) and (45), we have

$$L_0 = 0.31,\quad L_1 = -3.5,\quad L_2 = 1.75,\quad L_3 = -12.63,$$
$$K_1 = -0.51,\quad K_2 = -0.13,\quad K_3 = 0.60. \tag{55}$$

Substituting these values into (47) and using (42), we find

$$C = 0.37. \tag{56}$$

The value of the constant C obtained is less than the experimental value C_{exp} by a factor of 5. As shown in [4], $C_{exp} \approx 1.9$. This disagreement (if it is not a result of errors in the calculations) can be related to the approximation used for the function $\bar{\varphi}$ in the equation for the characteristic two-point function (see expression (7) in [1]). The following expression can be used as a corrected expression for $\bar{\varphi}$ together with the normalization condition:

$$\bar{\varphi} = \exp\{\lambda\,[B(0)\,\theta^{12} + [B_\beta(x) - B_\beta(z)]\,\theta_\alpha\theta_\beta] + \varkappa\,[G(0)\,\eta^{12} + [G(x) - G(z)]\,\eta\eta']\}, \tag{57}$$

where λ and $\varkappa$ are constants which are chosen so as to give the best agreement with experiment in the inertial and inertial-convective scale intervals. In order that C calculated from Eq. (47) be equal to 1.9, it is necessary to choose $\lambda \approx 1.815$. It is clear, however, that the correct expression for $\bar{\varphi}$ can only be obtained as a result of some variational calculation in the spirit of [6].

In conclusion, we made several remarks of a general nature. The purpose of this work and [1] and [2] was mainly to develop a technique for deriving equations for the structure function D(r, t), beginning with a closed equation for the characteristic function $\varphi_{\vec{r},t}(\vec{\theta}, \eta)$ of the joint probability density for differences in velocities and concentrations of a passive admixture at two points of an isotropic turbulent flow. In going over from the equation for φ to the equations for D, it is necessary to solve the closure problem, since the right side of the equation for φ includes, generally speaking, moments of all orders of the differences in velocities and concentrations at two points. The closure hypothesis used in [2] that the cumulants, beginning with the fourth and higher orders, are small is more of a working hypothesis than a recipe claiming a priori justification and finality. The guiding principles in choosing a closure scheme were considerations of relative simplicity and some experimental indications. The use of more complicated closure schemes, e.g., the scheme proposed in [6], or the choice of the form of the function φ, taking into account nonzero excess in the probability distribution, requires much more calculation than necessary for realizing the goals of the present work.

As is evident, the work carried out here has shown that the method proposed leads to equations for structure functions that differ greatly from the corresponding closed equations obtained with the use of the moment formalism. The nature of the nonlinearity turns out to be very complicated. In addition to the strong nonlinearity of the equations, which is manifested in their overall structure, there is also a weaker nonlinear dependence of the kernels of these integrodifferential equations on slowly varying combinations $\eta(y)$, $\beta(y)$, and $\gamma(y)$, which are determined by quantities of the type y[ln D(y)] etc.

Analysis of the stationary form of the equation in inertial scale interval shows that all integrals on its right side converge only if we solve the equation for the function $P(r) = {}^1/_2 D'_r(r)$, whose physical meaning is that of an energy density distribution in the scale space r, and not for D(r). In attempting to solve Eq. (27) for D(r), it turns out that the integral over ρ diverges. The situation here, apparently, is analogous to that encountered in Kreichnan's theory of direct interactions [7]. Sosinovich [8, 9] attempted to eliminate this difficulty by introducing into the theory some constants whose values where chosen from the requirements that the integral over ρ converge at infinity. However, the approach used in the present work is better justified, i.e., the transition to Eq. (30) for the function P(r), in which all integrals converge. This approach corresponds to Chandrasekhar's results [5], which confirm that only P(r) can be the object of study of the theory of isotropic turbulence in a universal scale interval.

NOTATION

D(r), two-point structure function of an isotropic turbulent velocity field; ε, specific pumping rate of turbulent energy; ε_d, specific rate of dissipation of turbulent energy; ν, kinematic viscosity coefficient; L, outer turbulent scale; C, constant in Kolmogorov's 2/3 law; $P(r) = {}^1/_2 D'_r(r)$.

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